Hence $N(th) = N\left(\frac{T}{T_B}\right)^{3/2}$

From

$$N = n_0 + N(th)$$

$$N = n_0 + N\left(T/T_B\right)^{3/2}$$

Get g/state population (condensation)

$$n_0 = N\left[1 - \left(T/T_B\right)^{3/2}\right]$$

Graphs of n_0 vs T } see sheet.
B vs T

Points

In 'condensation' temp range $0 < T \leq T_B$

(i) G/state occupation falls from N at $T=0$
to 0 at $T = T_B$

Many (n_0) particles carrying no energy — do not contribute to U, C_V, S

Condensed fraction in absolute zero state.

(ii) Total energy U given by bridge equation

$$\frac{U}{N(\text{th})} = \frac{\partial}{\partial \beta}(\ln Z)$$

where $Z = V\left(\frac{2\pi m k T}{h^2}\right)^{3/2} = \text{const.}\ T^{3/2}$

$$U = N(\text{th}).\ kT^2 \frac{\partial}{\partial T}\left(\ln\left(\text{const.}\ T^{3/2}\right)\right)$$

$$U = N\left(\frac{T}{T_B}\right)^{3/2} kT^2 .\ \frac{3}{2} \cdot \frac{1}{T}$$

$$U = \frac{3}{2}\frac{Nk}{T_B^{3/2}} .\ T^{5/2}$$

(iii) Heat Capacity $C_V = \left(\frac{\partial U}{\partial T}\right)_V$

$$C_V = \frac{15}{4} Nk \left(\frac{T}{T_B}\right)^{3/2}$$

(iv) Pressure

$P = \frac{2U}{3V}$ — see below.

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$$\downarrow \qquad\qquad \downarrow$$

$$U = N\left(\frac{T}{T_B}\right)^{3/2} kT^2 . \frac{3}{2} \cdot \frac{1}{T}$$

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$$C_V = \frac{15}{4} Nk \left(\frac{T}{T_B}\right)^{3/2}$$

(iv) Pressure

$$P = \frac{2U}{3V}$$ — see below.

Thus $p = \frac{Nk}{V} \frac{T^{5/2}}{T_B^{3/2}}$

These variations of U, C_V, P are for temperature region $0 < T < T_B$.

(v) In temperature region $T \gg T_B$ regain classical relations

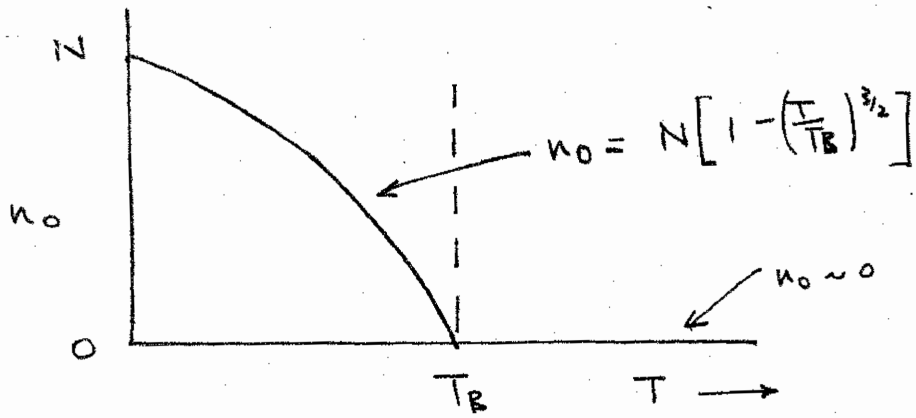
$$U = \frac{3}{2} NkT$$

$$C_V = \frac{3}{2} Nk$$

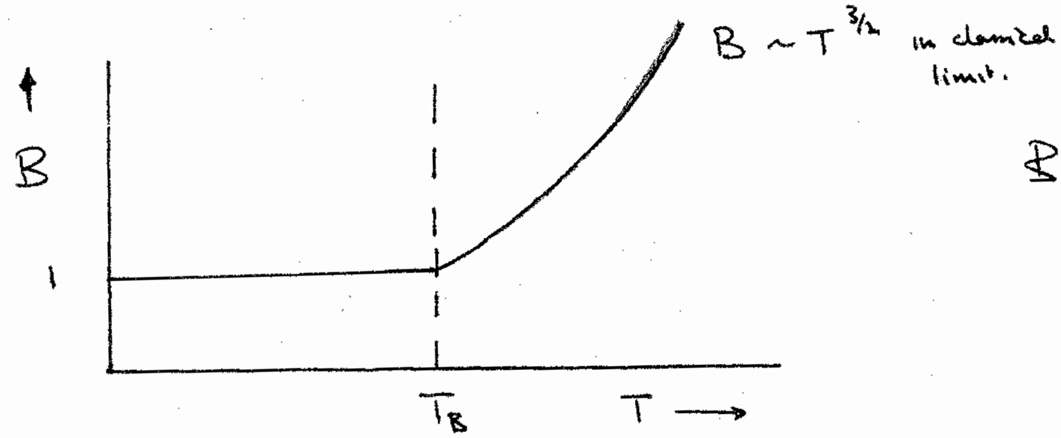
$$P = \frac{NkT}{V}$$

Variation of $\left.\begin{array}{l} U \\ C_V \\ P \end{array}\right\}$ vs T — see sheet.

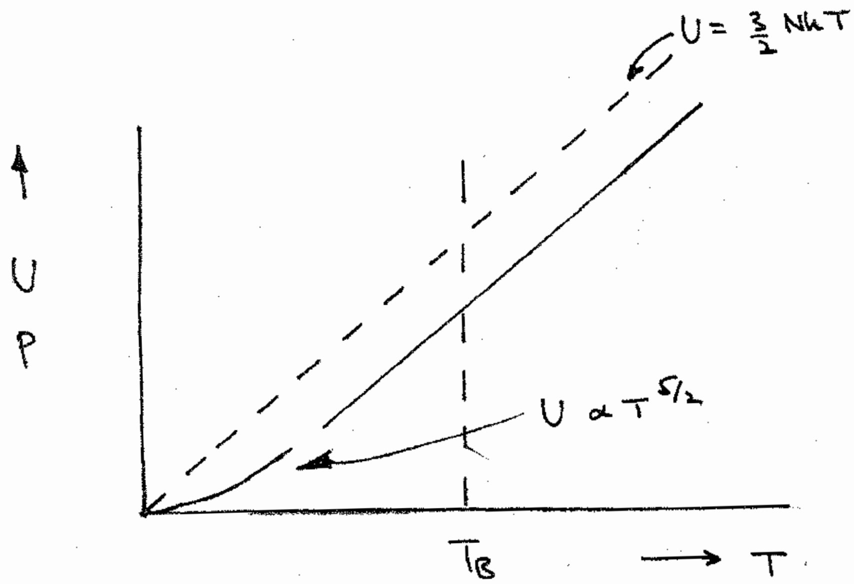
Boson particle gas.



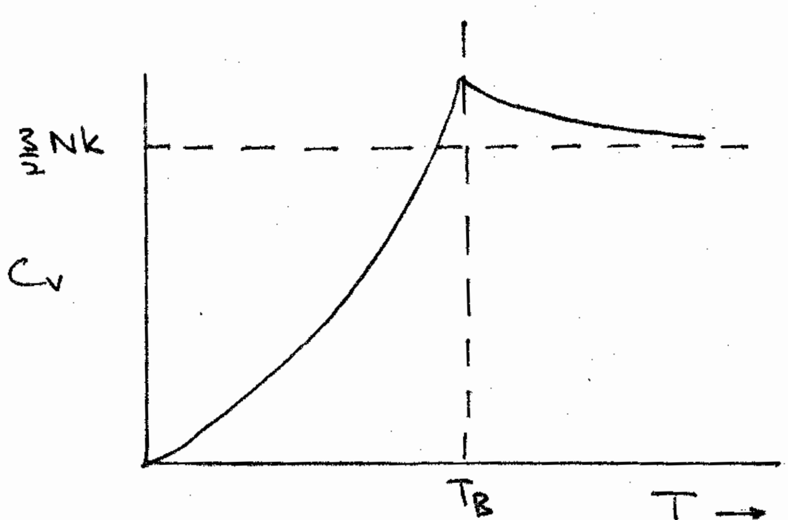
n_0 vs T



B vs T



U
P vs T



C_v vs T

Bose - Einstein condensation causes dramatic effects in properties of

(i) Liquid He^4

(ii) Superconducting materials

These discussed in Low Temperature part of course.

Massless boson systems

Photons – electromagnetic radiation

Phonons – quantised lattice waves in solids.

Photons.

Radiation in an enclosure (black box)

Want to derive the spectrum of radiation that is in equilibrium with enclosure of temperature T.

Spectrum is distribution $u(\nu)\,d\nu$ – the energy contained in radiation of frequency between ν and $\nu + d\nu$.

Want $u(\nu)$ as function of ν

Planck's radiation formula.

Treat photons in an enclosure as Bose - Einstein gas.

but N not constant — photons can be created / destroyed at walls.

Follow same B-E derivation but omit condition $\sum_i dn_i = 0$

so in previous case put $\alpha = 0$.

Get $$f(\epsilon) = \frac{1}{\left[\exp(\epsilon/kT) - 1\right]}$$

Derivation of $u(\nu)$

Photons are waves in a box.

Number of states with $k \to k+dk$ is

$$g(k)\,dk = \frac{V}{(2\pi)^3} \cdot 4\pi k^2 dk \times 2$$

2 polarisations.

and

Want distribution in terms of freq ν

Relation $\quad \nu = \frac{c}{\lambda} = \frac{kc}{2\pi}$

$$k^2\,dk = \left(\frac{2\pi}{c}\right)^3 \nu^2\,d\nu$$

Number of states in range $\nu \to \nu + d\nu$

$$g(\nu)\,d\nu = \frac{V}{(2\pi)^3} \cdot \left(\frac{2\pi}{c}\right)^3 \cdot 8\pi\,\nu^2\,d\nu$$

$$= \frac{8\pi V}{c^3}\,\nu^2\,d\nu$$

Energy distribution

$$u(\nu)\,d\nu = g(\nu)\,d\nu \,.\, f(\nu) \,.\, h\nu$$

$$= \frac{8\pi V}{c^3}\,\nu^2\,d\nu \,.\, \frac{1}{\left[\exp(h\nu/kT) - 1\right]} \,.\, h\nu$$

$$u(\nu)\,d\nu = \frac{8\pi V h\,\nu^3\,d\nu}{c^3\left[\exp(h\nu/kT) - 1\right]}$$

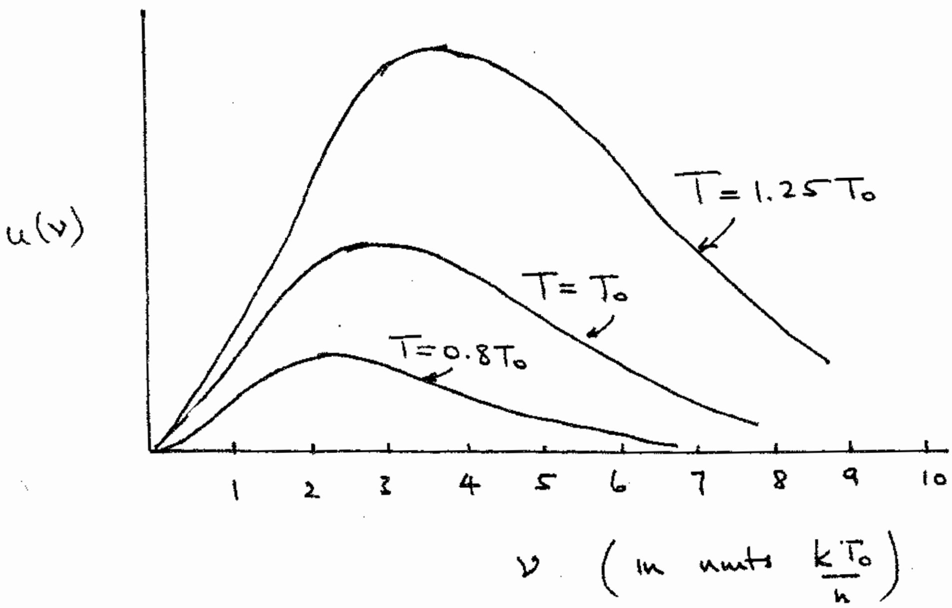
Planck radiation formula for black body

Plot - see

Planck radiation formula.

In black body enclosure at temperature T – distribution of energy u as function of photon frequency ν is

$$u(\nu) = \frac{8\pi V h \,.\, \nu^3}{c^3 \left[\exp(h\nu/kT) - 1 \right]}$$



Planck formula for $0.8 T_0$

T_0

$1.25 T_0$

Area under curve increases as T^4.

Points

(i) Excellent agreement with expt for $u(\nu)$.

(ii) For maxima in distributions at ν_{max} —

$\nu_{max} \propto T$. (~~Ween's~~ Wien's Law).

(iii) Total energy density $\left(\frac{U}{V}\right)$ is integral under curve.

$$\frac{U}{V} = \int_0^\infty u(\nu)\, d\nu$$

$$\frac{U}{V} = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3\, d\nu}{\left[\exp\left(h\nu/kT\right) - 1\right]}$$

Putting $y = \frac{h\nu}{kT}$ $\quad d\nu = \frac{kT\,dy}{h}$

$$\frac{U}{V} = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{y^3\, dy}{\left[\exp(y) - 1\right]}$$

$$= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \left(\frac{\pi^4}{15}\right)$$

integral

$$\frac{U}{V} = \frac{8\pi^5}{15}\left(\frac{1}{hc}\right)^3 (kT)^4 .$$

(iv) Stefan's Law.

Energy radiated $m^{-2} s^{-1}$ from black body $= \sigma T^4$.

Number of photons emitted $/ s.\, m^2 = \frac{1}{4}\frac{N}{V} . c$

— see sheet.

Energy emitted $/ m^2 s = \frac{1}{4}\left(\frac{N}{V}\right) c . \overline{h\nu}$

where $\overline{h\nu}$ is mean photon energy

Energy emitted $/ m^2 s = \frac{1}{4}\left(\frac{U}{V}\right) . c$

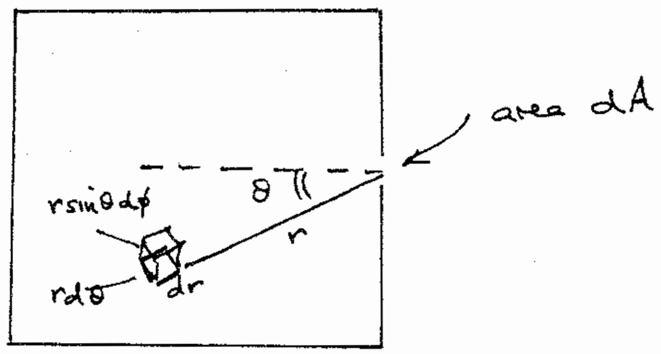
$$= \frac{1}{4} . c \left(\frac{8\pi h}{c^3}\right) \frac{\pi^4}{15} . \left(\frac{kT}{h}\right)^4$$

Excellent agreement with expt for

(i) variation as T^4

(ii) Value of σ

Number of photons striking area $1\,m^2$ / second.



Consider elemental box of side $dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi$ as shown.

Number of photons in this volume $= \left(\frac{N}{V}\right) \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Fraction that will go thru $dA = \frac{1}{4\pi} \cdot \frac{dA \cdot \cos\theta}{r^2}$

Number of photons from box passing thru dA

$$= \left(\frac{N}{V}\right) \cdot \frac{1}{4\pi} \cdot \frac{dA \cos\theta}{r^2} \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Total number of photons passing thru dA/s.

$$= \frac{dA}{4\pi} \cdot \left(\frac{N}{V}\right) \cdot \int_0^c dr \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{dA}{4\pi} \cdot \left(\frac{N}{V}\right) \quad c \quad \cdot \quad \frac{1}{2} \quad \cdot \quad 2\pi$$

$$= dA \cdot \frac{1}{4}\left(\frac{N}{V}\right) c$$

Number photons hitting $1\,m^2/s \quad = \quad \frac{1}{4}\left(\frac{N}{V}\right) c$.

(120)

(v) Radiation pressure

$$p = \frac{U}{3V}$$ — see sheet.

(vi) Photon theory — valid over what range of temperature?

Integration must be valid

Requires $kT > \Delta\epsilon$ — (energy state spacing)

For states in box of side a

$$\Delta\epsilon \sim \frac{hc}{a}$$

Limiting value of temperature T where

$$kT = \frac{hc}{a}$$

$$T = \frac{hc}{ka} \sim 1\,K \text{ for } a = 0.01\,m.$$

At temperature $T = 1\,K$ radiation not important in practice — theory OK.

Pressures.

Radiation pressure and pressure from particles with mass.

For both $$p = -\left(\frac{\partial U}{\partial V}\right)_S$$

$$p = -\frac{\partial}{\partial V}\left(\sum_i n_i \epsilon_i\right)$$

Radiation

$$\epsilon = \hbar c k$$

Massive particles

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$

Quantisation of k in box of side L

$$k = \frac{2\pi}{L} \cdot n = 2\pi n . V^{-1/3}$$

$$\epsilon = \text{const} \times V^{-1/3}$$

$$\frac{\partial \epsilon}{\partial V} = -\frac{1}{3}\left(\frac{\epsilon}{V}\right)$$

$$\epsilon = \text{const} \times V^{-2/3}$$

$$\frac{\partial \epsilon}{\partial V} = -\frac{2}{3}\left(\frac{\epsilon}{V}\right)$$

$$p = -\sum n_i \left(\frac{\partial \epsilon_i}{\partial V}\right)$$

$$p = +\frac{1}{3}\frac{U}{V}$$

pressure of photons in box

$$p = \frac{2}{3}\frac{U}{V}.$$

pressure of particles in box.

Phonons.

In a solid vibrating atoms interact strongly but can consider atomic motions as superposition of quantised lattice waves (phonons) which are weakly interacting

Lattice waves are bosons with distribution function $f(\epsilon) = \dfrac{1}{\exp(\epsilon/kT) - 1}$

Points

(i) For phonons degeneracy $G = 3$

representing transverse polarisations $\updownarrow$ and $\nwarrow\!\!\searrow$

and longitudenal waves $\longleftrightarrow$

(ii) Dispersion relation ϵ vs k

Depends on solid – monatomic

diatomic etc

but reasonable approximation at low ϵ and k is

$$\epsilon = \frac{hk}{2\pi} c_s \qquad c_s = \text{velocity of sound (phonons)}$$

(iii) For N atoms.

$3N$ vibrations in 3D.

Gives rise to just $3N$ quantised lattice waves (phonons).

Proper selection of waves needs details of solid but reasonable approx — Debye theory cuts off phonon density of states at freq ν_D where

$$3N = \int_0^{\nu_D} g(\nu)\, d\nu = \int_0^{\nu_D} \frac{12\pi V}{c_s^3} \nu^2\, d\nu$$

ν_D = Debye frequency

equivalent temp $\Theta_D = \dfrac{h\nu_D}{k}$. (Debye temp)

Debye theory vibration energy

$$U = \int_0^{\nu_D} g(\nu)\,d\nu \, . \, h\nu \, . \, f(\nu)$$

$g(\nu)\,d\nu$: density states $\nu \rightarrow \nu + d\nu$

$h\nu$: phonon energy

$f(\nu)$: prob state ~~with freq~~ ν is occupied

$$U = \int_0^{\nu_D} \frac{12\pi V \nu^2 d\nu}{c_s^3} \, . \, h\nu \, . \, \frac{1}{\left[\exp\left(h\nu/kT\right) - 1\right]}$$

put $y = \dfrac{h\nu}{kT}$ $\qquad y_D = \dfrac{h\nu_D}{kT}$

$$\nu = \frac{kTy}{h}$$

$$d\nu = \left(\frac{kT}{h}\right) dy$$

Substituting - get.

(iv) Debye theory vibration energy (see sheet)

$$U = V \cdot \left(\frac{12\pi h}{c_s^3} \right) \left(\frac{kT}{h} \right)^4 \int_0^{y_D} \frac{y^3 \, dy}{[\exp(y) - 1]}$$

where $y = \frac{h\nu}{kT}$

and $y_D = \frac{h\nu_D}{kT}$.

For high temperature $T \gg \theta_D$

$y \ll 1$

$$\exp(y) \sim 1 + y \ldots$$

$$\int \longrightarrow \int_0^{y_D} y^2 \, dy = \frac{1}{3} \left(\frac{h\nu_D}{kT} \right)^3$$

But from $3N = \int_0^{\nu_D} \frac{12\pi V}{c^3} \nu^2 \, d\nu$

get $\nu_D^3 = \frac{3}{4\pi} \left(\frac{N}{V} \right) c^3$

Gives $U = 3NkT.$

At low temperature $T \ll \Theta_D$

$$y_D \longrightarrow \infty$$

$$\int_0^\infty \frac{y^3\,dy}{\exp(y) - 1} \longrightarrow \frac{\pi^4}{15}$$

$$U = \frac{12\pi h V}{c_s^3}\left(\frac{kT}{h}\right)^4 \cdot \frac{\pi^4}{15}$$

Express this in terms of $\Theta_D = \dfrac{h\nu_D}{k}$

Via $\quad c_s^3 = \dfrac{4\pi}{3}\left(\dfrac{V}{N}\right)\nu_D^3 \quad$ (from above)

$$c_s^3 = \frac{4\pi}{3}\left(\frac{V}{N}\right)\left(\frac{k\Theta_D}{h}\right)^3$$

Substituting in expression for U gives

$$U = \frac{9\pi^4}{15} \cdot Nk\,\frac{T^4}{\Theta_D^3}$$

Heat capacity C_V.

$$C_V = \left(\frac{dU}{dT}\right)_V$$

At high temperature $C_V \to 3Nk$

At low temperature

$$C_V = \frac{d}{dT}\left(\frac{9\pi^4}{15} \cdot Nk \frac{T^4}{\Theta_D^3}\right)$$

$$C_V = \frac{12}{5}\pi^4 Nk \left(\frac{T}{\Theta_D}\right)^3$$

These predictions of Debye theory agree well with experiment.

At low temps Debye prediction

$$C_V \propto T^3$$

much better than Einstein model prediction

$$C_V \propto \exp\left(-\Theta_E/T\right)$$

Conduction electrons.

How well does Fermi Dirac gas explain properties of conduction electrons in metals?

Treats correctly

(i) Quantum energy states of particles in a box

(ii) Fermion statistics $f(\epsilon) = \dfrac{1}{\exp\left(\epsilon - \mu/_{kT}\right) + 1}$

Problem

Fermi-Dirac gas assumes weak interactions between particles.

Actually coulomb interactions of electrons

(i) with +ve ions of lattice

(ii) with other electrons

have energies $\simeq \mu$ (Fermi energy) – strong

Successes of F. D gas theory

Predicts correctly

(i) Size of Fermi energy $\mu \sim$ few e.V.

(ii) Heat capacity of conduction electrons

$$C_V = \gamma T$$

(iii) Magnetic susceptibility of conduction electrons

χ is (i) small

(ii) independent of T

Incorrect details

(i) Exact values of μ

(ii) In $C_V = \gamma T$

actual values of γ greater than F. D prediction

for	Na	by factor	1.2
	Al	" "	1.5
	Pb	" "	2.1

Inaccuracies of F.D gas predictions arise because of neglect of interactions

Surprise is that it does so well.

Reasons why strong interactions cause only small effects.

(i) Long range cancellation.

Metal is charge neutral.

At distance $r >$ few lattice spacings interaction of electron with

(i) +ve ions on lattice

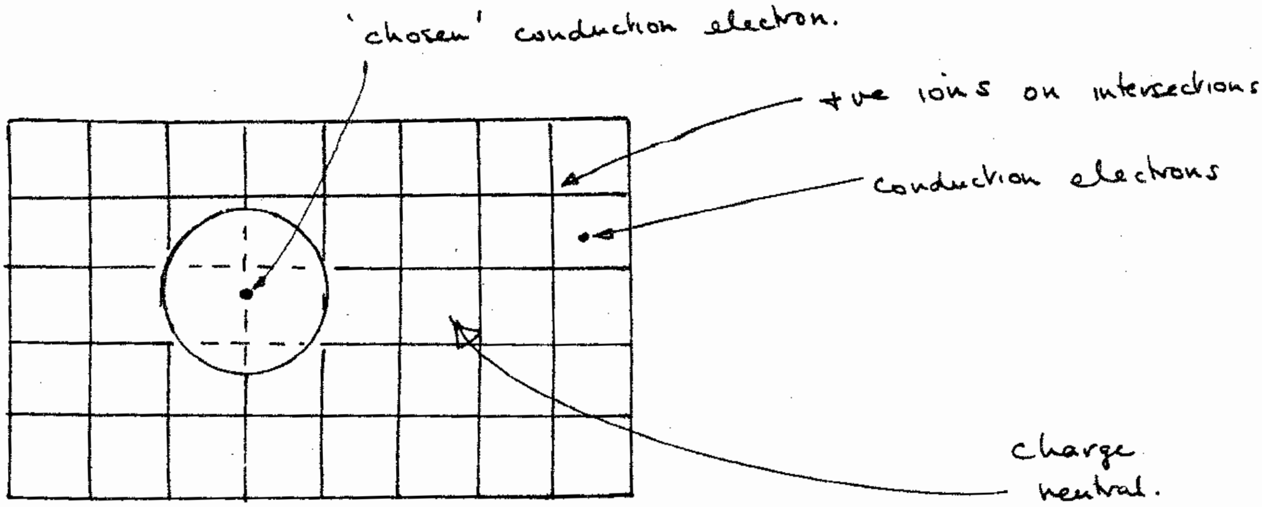
(ii) other conduction electrons

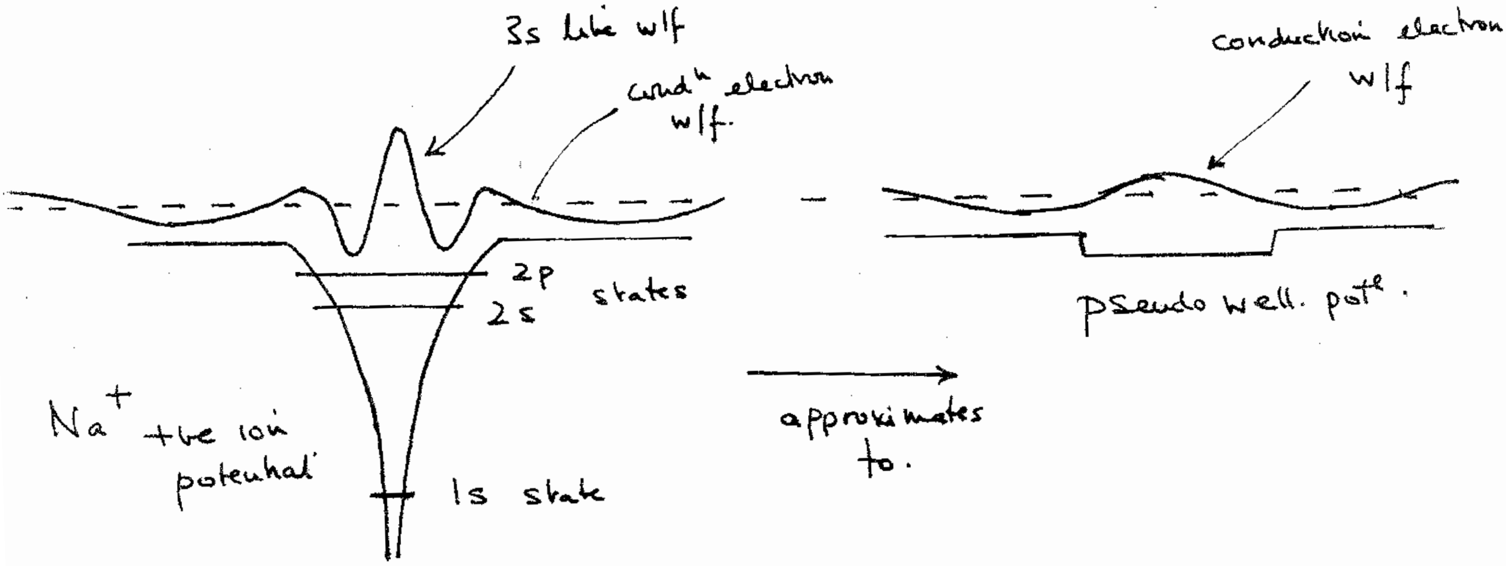
cancel out. — see picture on sheet.

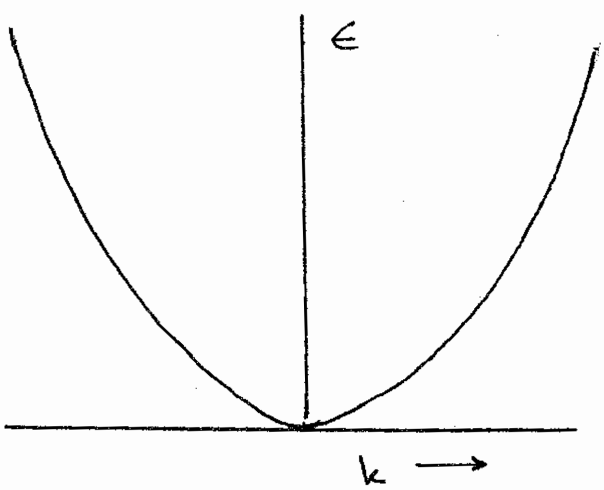
(ii) Pauli principle — pseudo potential model.

Electron close to +ve ion — sees it as deep potential well.

Conduction electron interactions

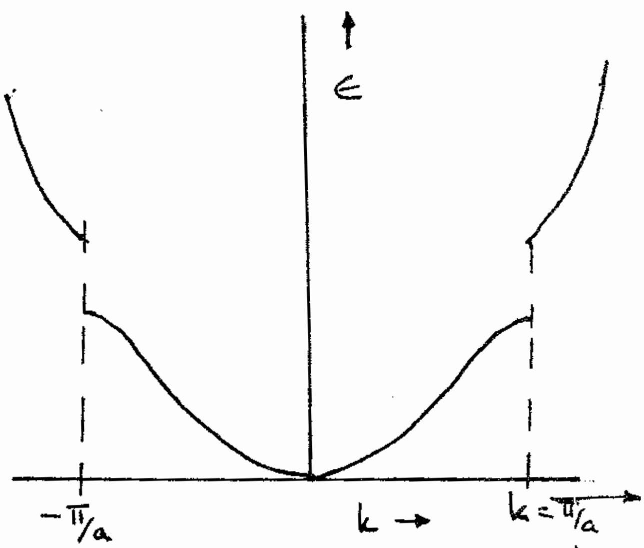






free electron ϵ vs k

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$



interacting electron ϵ vs k

Example Na metal.

Conduction electron feels pot^{l} well of Na^+ ion

Na^+ ion has bound state electrons $(1s)^2(2s)^2(2p)^6$

Pauli principle requires cond^n electron w/f

to be orthogonal to all bound state w/fs.

Result – cond^n electron w/f similar to

3s w/f (the ionised state)

This 3s w/f has high kinetic energy

High k.e in deep potential well – can

approximate to

Smaller k.e and shallow well

– See picture on sheet.

Approximation close to weakly interacting

electron.

Taking account of interactions

(I) Single particle approximation

Keep single particle theory

Effect of interactions change particle characteristics.

(Particle + effect of interactions) — called quasiparticle

Effect of interactions felt through ϵ vs k relation

Example.

Mass of quasiparticle $m^* = \dfrac{\hbar^2}{\left(\dfrac{d^2\epsilon}{dk^2}\right)}$

For free particle $\epsilon = \dfrac{\hbar^2 k^2}{2m}$

Then $m^* = m$

Quasiparticle mass can vary with

(i) particle energy